Unique continuation for the Helmholtz equation using a stabilized finite element method

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Based on a joint work with
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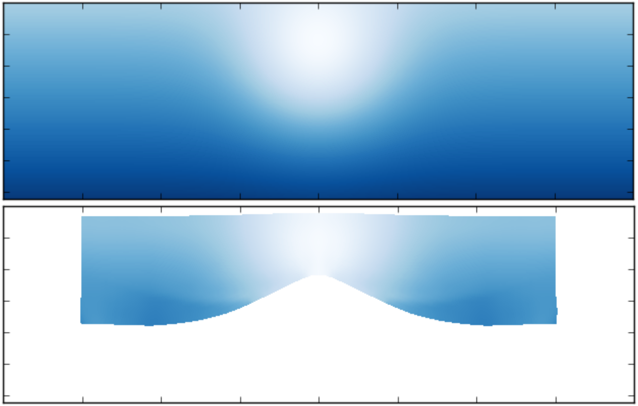
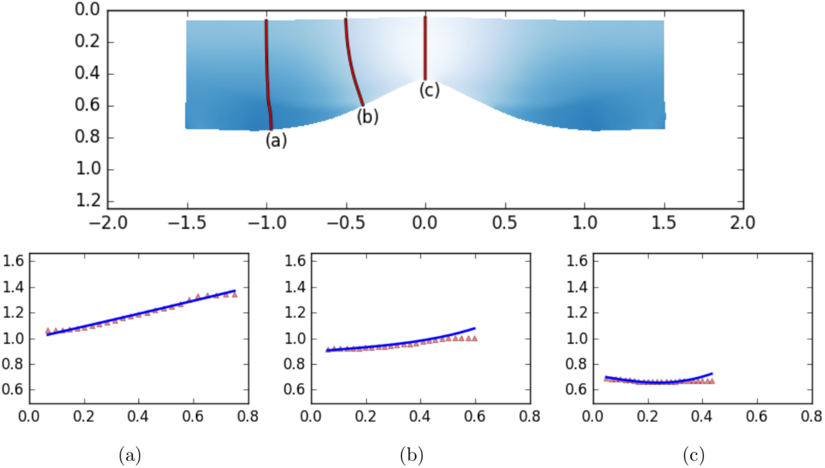
Motivation: recovering a speed of sound by layer stripping

Reconstruction of the front face of an acoustic lens using a variant of the Boundary Control method [DE HOOP-KEPLEY-L.O.].



We find the speed sound $c(x)$ in $\partial_t^2 u - c^2 \Delta u = 0$ given u and $\partial_\nu u$ on the boundary/surface for many solutions u. This is easiest near the boundary.

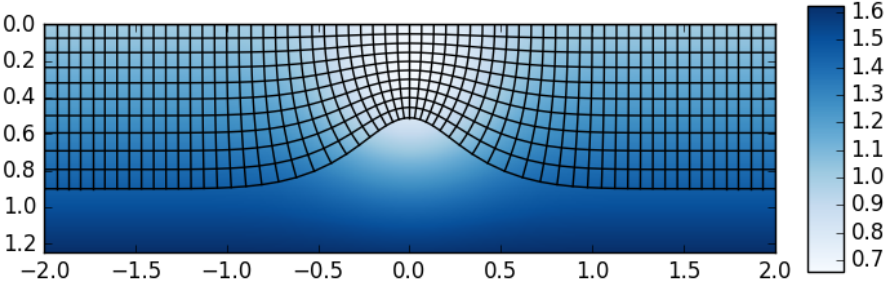
Motivation: recovering a speed of sound by layer stripping



True speed of sound (blue curve) and the reconstructed one (red triangles) as a function of depth along a ray path.

Boundary normal coordinates for the lens

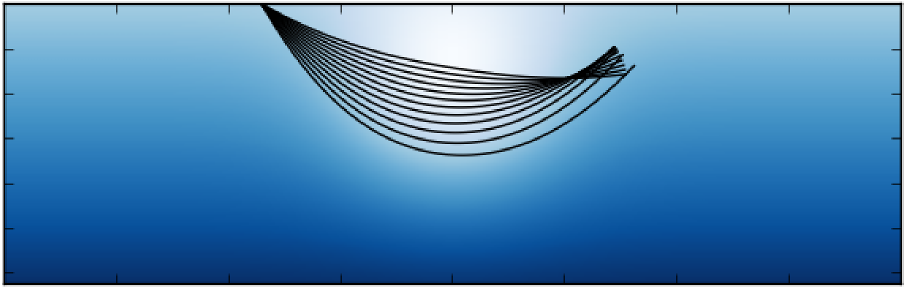
Reconstruction is computed on a rectangular patch in boundary normal coordinates.



The boundary normal coordinates degenerate behind the lens.

Focusing ray paths

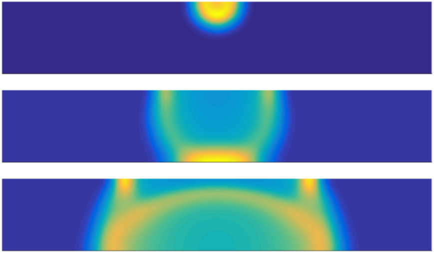
- In theory, the Boundary Control method avoids problems related to focusing by recovering the speed of sound in patches.
- The data needs to be continued across regions where the speed of sound is already known.

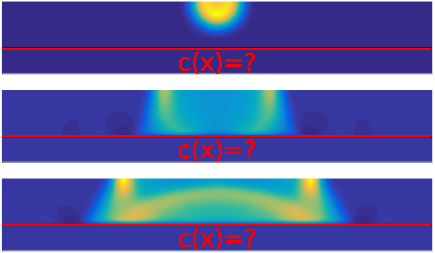


Some ray paths emanating from a point at the surface.

Unique continuation

- In theory, the data can be extended by using unique continuation.
- The rest of the talk focuses on numerical analysis of unique continuation in the *frequency domain*.

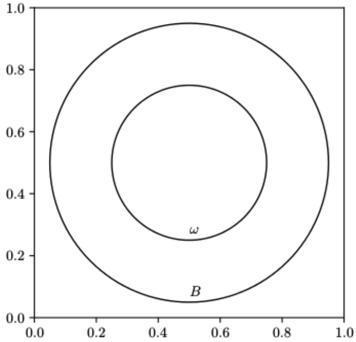




Left. Wave field that reflects at the bottom of a slab. *Right.* The same snapshots computed without knowing the speed of sound below the red line [de Hoop-Kepley-L.O.].

Unique continuation problem for the Helmholtz equation

Consider three open, connected and non-empty sets $\omega \subset B \subset \Omega$ in $\mathbb{R}^n$.



Unique continuation problem. Given $u|_\omega$ determine $u|_B$ for a solution u to the Helmholtz equation $\Delta u + k^2 u = 0$ in Ω.

Conditional Hölder stability/three solid balls inequality

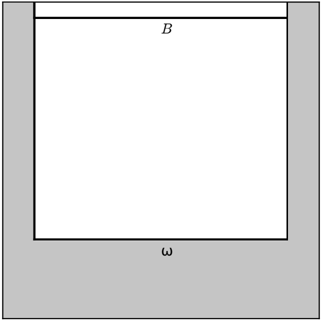
If B does not touch the boundary of Ω, then the unique continuation problem is conditionally Hölder stable.

For all $k \geq 0$ there are $C > 0$ and $\alpha \in (0, 1)$ such that

$$\|u\|_{H^1(B)} \leq C(\|u\|_{H^1(\omega)} + \left\|\Delta u + k^2 u\right\|_{L^2(\Omega)})^\alpha \|u\|_{H^1(\Omega)}^{1-\alpha}.$$

- In general, the constant C depends on k.
  - If there is a line that intersects B but not ω, then C blows up faster than any polynomial in k.
  - This can be shown by constructing a WKB solution localizing on the line (quasimode with non-homogeneous boundary conditions).
- Assuming suitable convexity, the constant C is independent of k.

Isakov's increased stability estimate



In a convex setting as above, it holds that

$$\|u\|_{L^2(B)} \leq CF + Ck^{-1} F^{\alpha} \|u\|_{H^1(\Omega)}^{1-\alpha},$$

where $F = \|u\|_{H^1(\omega)} + \left\|\Delta u + k^2 u\right\|_{L^2(\Omega)}$ and the constants C and α are independent of k.

Shifting in the Sobolev scale

Recall that $F = \|u\|_{H^1(\omega)} + \left\|\Delta u + k^2 u\right\|_{L^2(\Omega)}$. In Isakov's estimate,

$$\|u\|_{L^2(B)} \leq CF + Ck^{-1}F^{\alpha} \|u\|_{H^1(\Omega)}^{1-\alpha}, \tag{1}$$

the sides of the inequality are at different levels in the Sobolev scale.

For a plane wave $u(x) = e^{i\boldsymbol{k}x}$, with $|\boldsymbol{k}| = k$, it holds that

$$\|u\|_{H^1(\omega)} \sim (1+k) \|u\|_{L^2(\omega)}.$$

This suggest that the analogue of (1), with both the sides at the same level in the Sobolev scale, could be

$$\|u\|_{L^2(B)} \leq CkE + CE^{\alpha} \|u\|_{L^2(\Omega)}^{1-\alpha},$$

where $E = \|u\|_{L^2(\omega)} + \left\|\Delta u + k^2 u\right\|_{H^{-1}(\Omega)}$.

Shifting in the Sobolev scale

Recall that $E = \|u\|_{L^2(\omega)} + \left\|\Delta u + k^2 u\right\|_{H^{-1}(\Omega)}$. We show a stronger estimate than

$$\|u\|_{L^2(B)} \leq CkE + CE^{\alpha} \|u\|_{L^2(\Omega)}^{1-\alpha}.$$

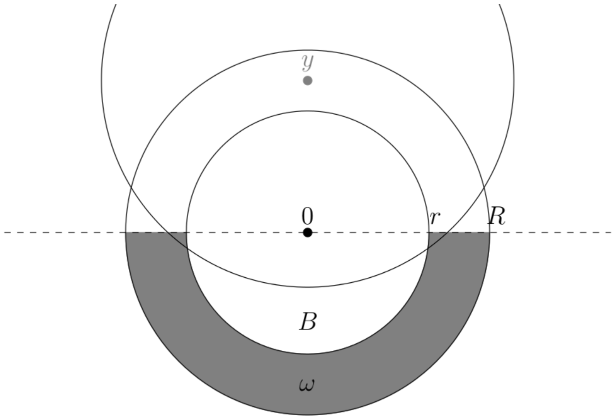
Lemma [Burman-Nechita-L.O]. For a suitable convex geometry $\omega \subset B \subset \Omega$. There are $C > 0$ and $\alpha \in (0, 1)$ such that for all $k \in \mathbb{R}$

$$\|u\|_{L^2(B)} \leq CE^{\alpha} \|u\|_{L^2(\Omega)}^{1-\alpha}.$$

Our numerical analysis is based on this estimate.

On the convexity assumption

We prove the estimate only in the particular geometry



This is a model for a local problem near a point on ∂B assuming that ∂B is convex there. In what follows, we will consider only this geometry.

Stabilized finite element method

We use the shorthand notation

$$G(u,z) = (\nabla u, \nabla z) - k^2(u,z),$$
$$(\cdot,\cdot) = (\cdot,\cdot)_{L^2(\Omega)}, \quad \|\cdot\|_\omega = \|\cdot\|_{L^2(\omega)}.$$

The stabilized FEM for the unique continuation problem is based on finding the critical point of the Lagrangian functional

$$L_q(u,z) = \frac{1}{2}\|u-q\|_\omega^2 + \frac{1}{2}s(u,u) - \frac{1}{2}s^*(z,z), + G(u,z),$$

on a scale of finite element spaces $\mathbb{V}_h$, $h > 0$. Here $q \in L^2(\omega)$ is the data.

The crux of the method is to choose suitable regularizing terms $s(u,u)$ and $s^*(z,z)$. They are defined only in the *finite element spaces*.

Error estimates

For a suitable choice of a scale of finite element spaces $\mathbb{V}_h$, $h > 0$, and regularizing terms $s(u, u)$ and $s^*(z, z)$, we show that

$$L_q(u, z) = \frac{1}{2}\|u - q\|_\omega^2 + \frac{1}{2}s(u, u) - \frac{1}{2}s^*(z, z) + G(u, z),$$

has a unique critical point $(u_h, z_h) \in \mathbb{V}_h$, and that for all $k, h > 0$, satisfying $kh \leq 1$, it holds that

$$\|u - u_h\|_{L^2(B)} \leq Ch^\alpha k^{2\alpha-2} \left(\|u\|_{H^2(\Omega)} + k^2 \|u\|_{L^2(\Omega)}\right).$$

Here u is the solution to the unique continuation problem

$$\begin{cases} \Delta u + k^2 u = 0, \\ u|_\omega = q. \end{cases}$$

Error estimates with noisy data

Consider now the case that $u|_\omega = q$ is known only up to an error $\delta q \in L^2(\omega)$. That is, we assume that $\tilde{q} = q + \delta q$ is known.

Let $(u_h, z_h) \in \mathbb{V}_h$ be the minimizer of the perturbed Lagrangian $L_{\tilde{q}}$. Then for all $k, h > 0$, satisfying $kh \leq 1$, it holds that

$$\|u - u_h\|_{L^2(B)} \leq C h^\alpha k^{2\alpha-2} \left(\|u\|_{H^2(\Omega)} + k^2 \|u\|_{L^2(\Omega)} + h^{-1} \|\delta q\|_{L^2(\omega)} \right),$$

where u is again the solution to the unique continuation problem

$$\begin{cases} \Delta u + k^2 u = 0, \\ u|_\omega = q. \end{cases}$$

On previous literature

Several authors, e.g. Bourgeois, Klibanov, ..., have considered the unique continuation problem for the Helmholtz equation from the computational point of view.

- They use the quasi-reversibilty method originating from [Lattès-Lions'67]
- No rate of convergence with respect to the mesh size is proven

For related problems, there are also methods based on Carleman estimates on discrete spaces e.g. by Le Rousseau.

Stabilized finite element methods for unique continuation, with proven convergence rates, have been recently developed in the following cases

- Laplace equation [Burman'14]
- Heat equation [Burman-L.O.]

Details of the finite element method

Let us now specify s and s^* and the domain $\mathbb{V}_h$ for the Lagrangian

$$L_q(u, z) = \frac{1}{2}\|u - q\|_\omega^2 + \frac{1}{2}s(u, u) - \frac{1}{2}s^*(z, z), + G(u, z).$$

Let V_h be the H^1-conformal approximation space based on the $\mathbb{P}_1$ finite element over a suitable triangulation of Ω. Here h is the mesh size. Set

$$\mathbb{V}_h = V_h \times W_h, \quad W_h = V_h \cap H_0^1(\Omega).$$

Denote by $\mathcal{F}_h$ the set of internal faces of the triangulation, and define

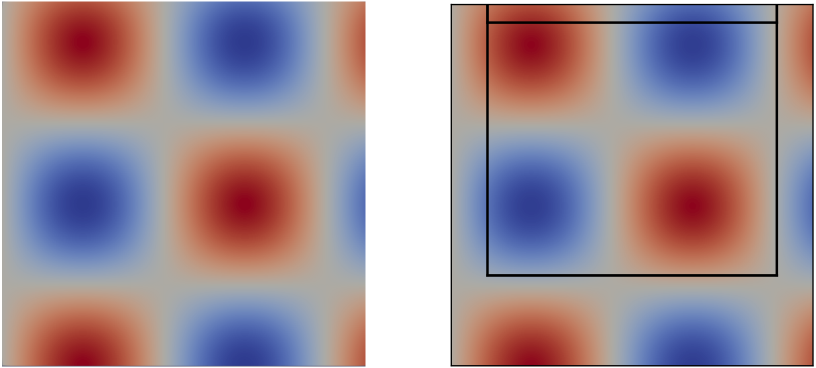
$$J(u, u) = \sum_{F \in \mathcal{F}_h} \int_F h [\![n \cdot \nabla u]\!]_F^2 \, ds, \quad u \in V_h,$$

where $[\![n \cdot \nabla u]\!]_F$ is the jump of the normal derivative. Set $\gamma = 10^{-5}$ and

$$s(u, u) = \gamma J(u, u) + \gamma \left\| hk^2 u \right\|_{L^2(\Omega)}^2, \qquad s^*(z, z) = \|\nabla z\|_{L^2(\Omega)}^2.$$

Computational example: a convex case

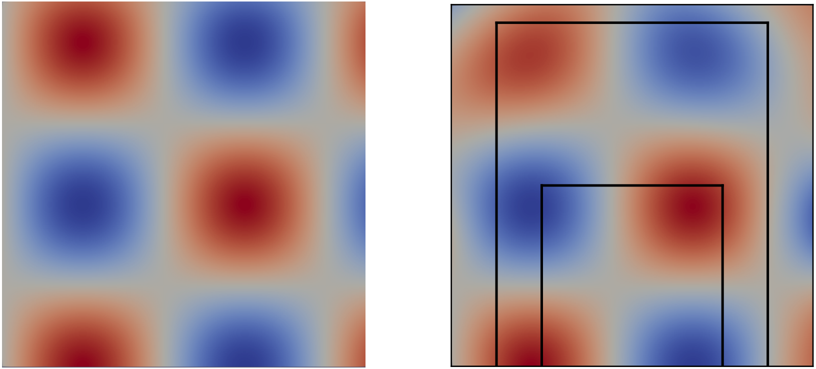
The unique continuation problem for the Helmholtz equation in the unit square with $k = 10$. The exact solution is $u(x, y) = \sin \frac{kx}{\sqrt{2}} \cos \frac{ky}{\sqrt{2}}$.
We use a regular mesh with $2 \times 256 \times 256$ triangles.



Left. True u. *Right.* Minimizer u_h of the Lagrangian L_q. Here ω is the region touching left, bottom and right sides.

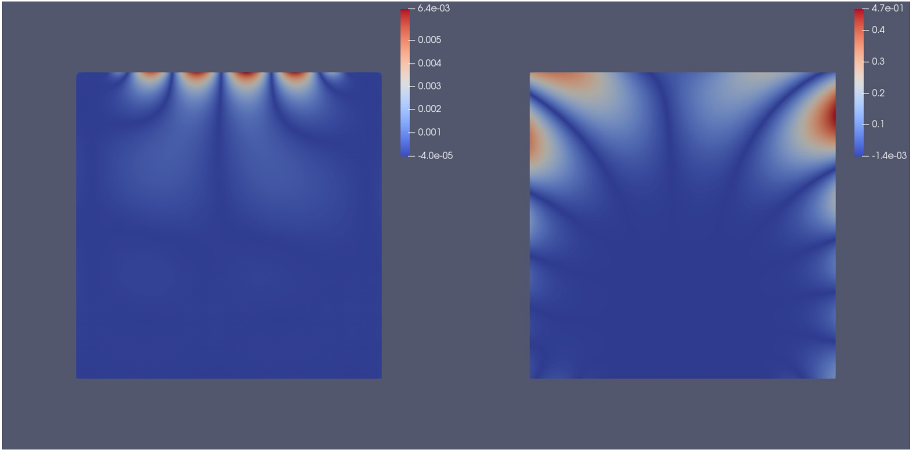
Computational example: a non-convex case

The same example except that ω is changed.



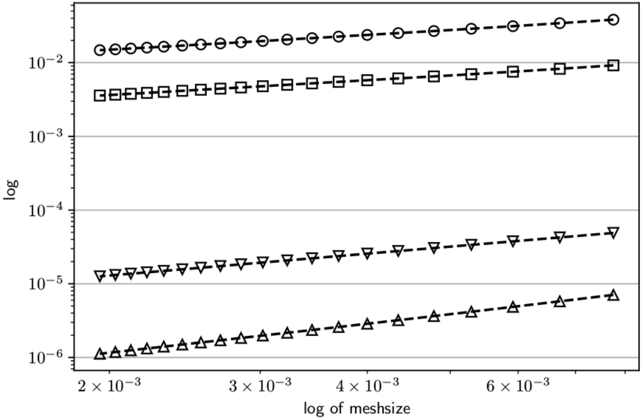
Left. True u. *Right.* Minimizer u_h of the Lagrangian L_q. Here ω is the rectangular region touching only the bottom side.

Comparison of the errors



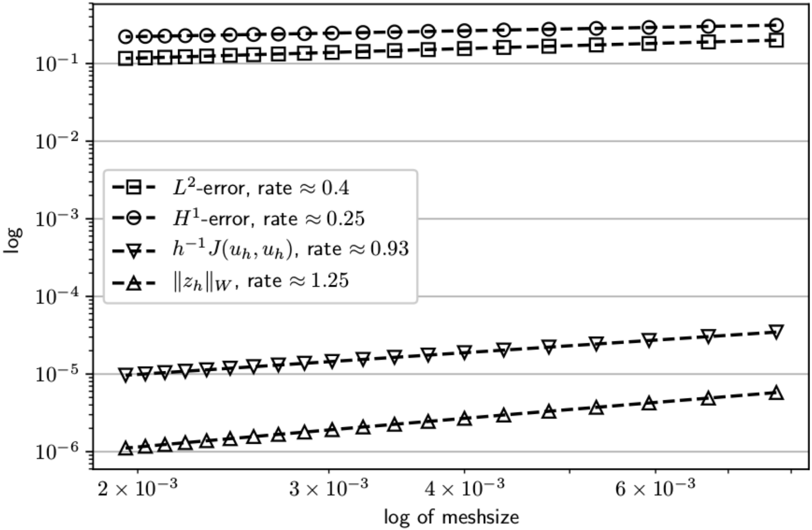
Left. Convex case *Right.* Non-convex case. Note that the scales differ by two orders of magnitude.

Convergence: the convex case

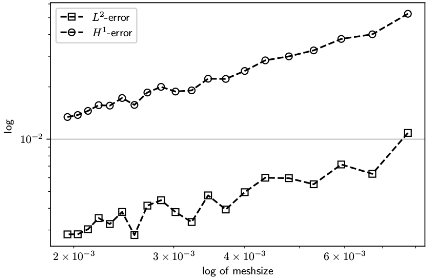


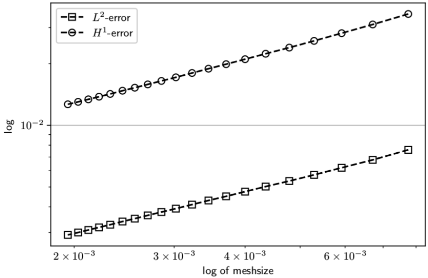
Circles: H^1-error, rate ≈ 0.64; squares: L^2-error, rate ≈ 0.66; down triangles: $h^{-1}J(u_h, u_h)$, rate ≈ 1; up triangles: $s^*(z_h, z_h)^{1/2}$, rate ≈ 1.3.

Convergence: the non-convex case



Convergence: the effect of noise in the convex case





Left. Perturbation $\mathcal{O}(h)$. *Right.* Perturbation $\mathcal{O}(h^2)$.

Ideas towards the proof in the case $k = 0$

Consider the Lagrangian

$$L_q(u, z) = \frac{1}{2}\|u - q\|_\omega^2 + \frac{1}{2}s(u, u) - \frac{1}{2}s^*(z, z), +(\nabla u, \nabla z),$$

on the discrete space $V_h \times W_h$, $W_h = V_h \cap H_0^1(\Omega)$, with

$$s(u, u) = J(u, u) = \sum_{F \in \mathcal{F}_h} \int_F h[\![n \cdot \nabla u]\!]_F^2 \, ds, \quad s^*(z, z) = \|\nabla z\|^2 .$$

The critical points (u, z) of L_q satisfy the normal equations

$$D_u L_q v = 0, \quad D_z L_q w = 0,$$

for all $v \in V_h$ and $w \in W_h$.

Reformulation of the normal equations

The normal equations

$$D_u L_q v = 0, \quad D_z L_q w = 0, \tag{2}$$

can be rewritten as $A[(u,z),(v,w)] = (q,v)_\omega$. Here

$$\begin{aligned}
L_q &= \frac{1}{2}\|u-q\|_\omega^2 + \frac{1}{2}s(u,u) - \frac{1}{2}s^*(z,z), +(\nabla u, \nabla z),\\
D_u L_q v &= (u-q,v)_\omega + s(u,v) + (\nabla v, \nabla z),\\
D_z L_q w &= -s^*(z,w) + (\nabla u, \nabla w),\\
A[(u,z),(v,w)] &= (u,v)_\omega + s(u,v) + (\nabla v, \nabla z) - s^*(z,w) + (\nabla u, \nabla w).
\end{aligned}$$

The well-posedness of (2) follows from the weak coercivity

$$|||(u,z)||| \le C \sup_{(v,w)\in V_h \times W_h} \frac{A[(u,z),(v,w)]}{|||(v,w)|||}.$$

Weak coercivity

Recall that $s(u,u) = J(u,u)$ and $s^*(z,z) = \|\nabla z\|^2$. We have

$$
\begin{aligned}
A[(u,z),(u,-z)] &= (u,u)_\omega + s(u,u) + (\nabla u, \nabla z) + s^*(z,z) - (\nabla u, \nabla z) \\
&= \|u\|_\omega^2 + J(u,u) + \|\nabla z\|^2 \\
&=: |||(u,z)|||^2.
\end{aligned}
$$

By Poincaré's inequality $\|\nabla z\|$ is a norm on $W_h = V_h \cap H_0^1(\Omega)$.

Lemma. $|||u|||^2 := \|u\|_\omega^2 + J(u,u)$ is a norm on V_h.

Proof. If $|||u||| = 0$ then $u = 0$ on ω. As V_h is H^1 conformal, u can not have a tangential jump across a face. As $J(u,u) = 0$, u can not have a normal jump across a face. Thus u can not change across a face, and therefore $u = 0$ identically.

Very rough sketch of the convergence

Let (u_h, z_h) be the critical point of the Lagrangian L_q where $q = u|_\omega$ and $\Delta u = 0$. Define the residual

$$r = \Delta(u_h - u).$$

The Hölder stability estimate gives

$$\|u_h - u\|_{L^2(B)} \leq C(\|u_h - u\|_{L^2(\omega)} + \|r\|_{H^{-1}(\Omega)})^\alpha \|u_h - u\|_{L^2(\Omega)}^{1-\alpha}.$$

- $\|u_h - u\|_{L^2(\omega)} + \|r\|_{H^{-1}(\Omega)} \leq Ch \|u\|_{H^2(\Omega)}$ follows from interpolation and weak coercivity, similarly with the usual FEM convergence.
- $\|u_h - u\|_{L^2(\Omega)} \leq C \|u\|_{H^2(\Omega)}$ follows from (interpolation, weak coercivity and) the quantitative version of the previous lemma:

Lemma. $\|u\|_{L^2(\Omega)} \leq Ch^{-1}(\|u\|_\omega^2 + J(u,u))$ for $u \in V_h$.

Convergence

Let (u_h, z_h) be the critical point of the Lagrangian L_q where

$$\begin{cases} \Delta u = 0, \\ q = u|_\omega. \end{cases}$$

Then

$$\|u_h - u\|_{L^2(B)} \leq Ch^\alpha \|u\|_{H^2(\Omega)}.$$

We emphasize that α is the exponent in the continuum stability estimate.